

# New attractor mechanism for spherically symmetric extremal black holes

Yun Soo Myung<sup>1,a</sup>, Yong-Wan Kim<sup>1,b</sup>, and Young-Jai Park<sup>2,c</sup>

<sup>1</sup>Institute of Mathematical Science and School of Computer Aided Science,  
Inje University, Gimhae 621-749, Korea

<sup>2</sup>Department of Physics and Center for Quantum Spacetime, Sogang  
University, Seoul 121-742, Korea

## Abstract

We introduce a new attractor mechanism to find the entropy for spherically symmetric extremal black holes. The key ingredient is to find a two-dimensional (2D) dilaton gravity with the dilaton potential  $V(\phi)$ . The condition of an attractor is given by  $\nabla^2\phi = V(\phi_0)$  and  $\bar{R}_2 = -V'(\phi_0)$  and for a constant dilaton  $\phi = \phi_0$ , these are also used to find the location of the degenerate horizon  $r = r_e$  of an extremal black hole. As a nontrivial example, we consider an extremal regular black hole obtained from the coupled system of Einstein gravity and nonlinear electrodynamics. The desired Bekenstein-Hawking entropy is successfully recovered from the generalized entropy formula combined with the 2D dilaton gravity, while the entropy function approach does not work for obtaining this entropy.

PACS numbers: 04.70.Bw, 04.70.Dy, 04.20.Dw, 04.20.Jb

Keywords: Entropy function; Regular black hole; Wald's Noether Charge; 2D Dilaton gravity.

<sup>a</sup>ysmyung@inje.ac.kr

<sup>b</sup>ywkim65@gmail.com

<sup>c</sup>yjpark@sogang.ac.kr

# 1 Introduction

Still growing interest in extremal black holes is motivated by their unusual and not fully understood nature. The problems of entropy, semiclassical configurations, interactions with matter, and information paradox have not been resolved completely. Apart from their global structure and behavior, the near horizon region is also of interest [1].

In particular, of the equal importance is the question of the singularities that reside in centers of most black holes hidden to an external observer. Regular black holes (RBHs) have been considered, dating back to Bardeen [2], for avoiding the curvature singularity beyond the event horizon [3]. Their causal structures are similar to a Reissner-Nordström (RN) black hole with the singularity replaced by de Sitter space-time [4]. In addition to various RBHs [5], the action of Einstein gravity and nonlinear electrodynamics provided a magnetically charged RBH [6]. This solution is featured by two integration constants and a free parameter. The integration constants are related to Arnowitt-Deser-Misner (ADM) mass  $M$  and a magnetic charge  $Q$ , while the free parameter  $a$  is adjusted to make the line element regular at center. Moreover, it allows exact treatment by using the Lambert function [7]. Here we note that this extremal RBH has the near horizon geometry of  $AdS_2 \times S^2$  as the extremal RN black hole does have [8, 9].

On the other hand, string theory suggests that higher curvature terms can be added to Einstein gravity [10]. Black holes in higher-curvature gravity [11] were extensively studied, showing the spectacular progress in the microscopic counting of black hole entropy. For a review, see [12]. In theories with higher curvature corrections, classical entropy deviates from the Bekenstein-Hawking entropy and can be calculated using Wald's Noether charge formalism [13]. It exhibits exact agreement with string theory predictions, both in the BPS [14] and non-BPS [15, 16] cases.

Recently, Sen has proposed a so-called “entropy function” method for calculating the entropy of  $n$ -dimensional extremal black holes, which is effective even for the presence of higher curvature terms. Here the extremal black holes are characterized by the near horizon geometry of  $AdS_2 \times S^{n-2}$  and corresponding isometry [17]. It states that the entropy of extremal black holes can be obtained by extremizing the entropy function with respect to some moduli on the horizon. Extremizing a entropy function is equivalent to solving Einstein equation in the near horizon. A entropy function usually depends only on the near horizon geometry, and decouples from the data at

infinity. This describes the attractor behavior. This method has been applied to many solutions including extremal black holes in higher dimensions, rotating black holes and various non-supersymmetric black holes [18]. We note that the near horizon isometry  $SO(2,1)$  and the long throat of  $AdS_2$  sector are the two ingredients of the attractor mechanism [19]. On the other hand, Cai and Cao [20] have proposed the generalized entropy formula based on Wald's Noether charge formalism. However, this generalized entropy function approach has a drawback that there is no way to combine the full equations of motion with the attractor mechanism.

Very recently, we have investigated a magnetically charged RBH [21]. It turned out that the entropy function approach does not work for deriving the Bekenstein-Hawking entropy of the extremal RBH, while the generalized entropy formula is suitable for the RBH case. This is mainly because the entropy function depends on the near horizon geometry ( $Q^2$ ) nonlinearly as well as the data at infinity ( $M$ ).

In this paper, we address this issue of the extremal RBH again. We introduce a new attractor mechanism to find Bekenstein-Hawking entropy for the extremal RBH. The important point is to find a 2D dilaton gravity with dilaton potential  $V(\phi)$  by imposing the dimensional reduction of 4D Einstein gravity including matter and then by performing a conformal transformation [22, 23]. Then, the new attractor equations are given by  $\nabla^2\phi = V(\phi)$  and  $V'(\phi) = -\bar{R}_2$ . For a constant dilaton  $\phi = \phi_0$ , this can be used to find the location  $r = r_e$  of degenerate horizon. Finally, we use the generalized entropy formula based on Wald's Noether charge formalism to derive the desired Bekenstein-Hawking entropy.

The organization of this work is as follows. In Sec. 2, we show the procedure to find a 2D dilaton gravity from a 4D Einstein gravity coupled with matter. In order to test whether this approach works for calculating the entropy of an extremal black hole, we study a toy model of the RN black hole in Sec. 3. We briefly review a magnetically charged RBH in Sec. 4. In Sec. 5, we show that Sen's entropy function approach does not work for the regular black hole. Sec. 6 is devoted to finding the entropy of an extremal RBH by using the generalized entropy function approach. We obtain the Bekenstein-Hawking entropy of an extremal RBH by using the 2D dilaton gravity approach with a conformal transformation in Sec. 7. Finally, we discuss our results in Sec. 8.

## 2 Dimensional Reduction Approach

We start with the four-dimensional (4D) action

$$I = \frac{1}{16\pi} \int d^4x \sqrt{-g} [R - \mathcal{L}_M(B)] \quad (1)$$

where  $\mathcal{L}_M(B)$  is the Lagrangian for matter. For our purpose, we consider the spherically symmetric metric

$$ds^2 = -U(r)dt^2 + \frac{1}{U(r)}dr^2 + b^2(r)d\Omega_2^2, \quad (2)$$

where  $b(r)$  plays a role of the radius of two sphere  $S^2$ . The 4D Ricci scalar  $R$  is calculated as

$$R = -U'' - \frac{1}{b^2} [4bb'U' + 2U(b'^2 + 2bb'') - 2], \quad (3)$$

where the prime denotes the derivative with respect to  $r$ . After the dimensional reduction by integrating the action (1) over  $S^2$ , the reduced effective action in two dimensions [22] can be rewritten by

$$I^{(2)} = \frac{1}{4} \int d^2x \sqrt{-g} [b^2 R_2 + 2g^{\mu\nu} \nabla_\mu b \nabla_\nu b + 2 - 4b^2 \mathcal{L}_M], \quad (4)$$

where  $R_2 = -U''(r)$  is the 2D Ricci scalar. It is convenient to eliminate the kinetic term by using the conformal transformation

$$\bar{g}_{\mu\nu} = \sqrt{\phi} g_{\mu\nu}, \quad \phi = \frac{b^2(r)}{4}. \quad (5)$$

This transformation delivers information on the 4D action (1) to 2D dilaton potential, if the 4D action provides the black hole solution. That is, we may get the good  $s$ -wave approximation to the 4D black hole eliminating the kinetic term. Unless one makes the conformal transformation, the information is split into the kinetic and the potential terms.

Now, let us choose the dilaton as the squared radius of  $S^2$  ( $\phi = r^2/4$ ). Then, the reparameterized action takes the form

$$\bar{I}^{(2)} = \int d^2x \sqrt{-\bar{g}} [\phi \bar{R}_2 + V(\phi)] \equiv \int d^2x \sqrt{-\bar{g}} \bar{F}, \quad (6)$$

where the Ricci scalar and the dilaton potential are given by

$$\bar{R}_2 = -\frac{U''}{\sqrt{\phi}}, \quad V(\phi) = \frac{1}{2\sqrt{\phi}} - \sqrt{\phi} \mathcal{L}_M(B), \quad (7)$$

respectively. This is a 2D dilaton gravity (for a review, see [24]) with  $G_2 = 1/2$  [25]. Also  $\bar{F}$  will play a role of the entropy function. The two equations of motion are derived as

$$\nabla^2 \phi = V(\phi), \quad (8)$$

$$\bar{R}_2 = -V'(\phi), \quad (9)$$

where  $V'(\phi)$  denotes the derivative with respect to  $\phi$ . These equations will play the role of a new attractor. Note that in the case of nonsupersymmetric attractors in four dimensions [15], the condition for attractor are that  $\partial_i V_{eff}(\phi_{i0}) = 0$  and  $\partial_i \partial_j V_{eff}(\phi_{i0})$  should have positive eigenvalues at the critical point of  $\phi_i = \phi_{i0}$ . If these conditions are satisfied, the attractor mechanism works and the entropy is given by the effective potential at horizon.

For our case, the corresponding conditions for the attractor are

$$V(\phi_0) = 0, \quad V'(\phi_0) \neq 0 \quad (10)$$

for the constant dilaton  $\phi = \phi_0$ . A solution to the equation (8) provides a constant dilation  $\phi_0$  when  $V(\phi_0) = 0$ . Considering the connection of  $\phi_0 = r_e^2/4$ , the solution to the equation (9) gives us information on the 2D spacetime

$$\bar{R}_2|_{r=r_e} = -\frac{U''(r_e)}{\sqrt{\phi_0}} \quad (11)$$

which is a constant curvature of the  $AdS_2$  spacetime.

After the conformal transformation, the generalized entropy formula derived by Wald's Noether charge formalism is obtained as

$$\bar{S}_{BH} = \frac{4\pi\sqrt{\phi_0}}{U''(r_e)}(qe - \bar{F}), \quad (12)$$

which is slightly different from the entropy formula  $S_{BH} = \frac{4\pi}{U''(r_e)}(qe - F)$ , proposed by Cai and Cao [20]. Finally, we have the generalized entropy function

$$\bar{F}(\phi_0) = -\sqrt{\phi_0}U''(r_e) \quad (13)$$

at the degenerate horizon. Considering  $\phi_0 = \frac{1}{4}r_e^2$ , the desired Bekenstein-Hawking entropy is obtained by

$$\bar{S}_{BH} = -\frac{4\pi\sqrt{\phi_0}}{U''(r_e)}\bar{F}(\phi_0) = 4\pi\phi_0 = \pi r_e^2 \quad (14)$$

for a magnetically charged black hole with  $e = 0$ .

### 3 Reissner-Norström black hole

We consider a toy model of Einstein-Maxwell theory to test whether our approach does work for obtaining a proper entropy of extremal RN black hole. In this case, we have

$$\mathcal{L}_M(B) = F_{\mu\nu}F^{\mu\nu} = \frac{2Q^2}{r^4} \quad (15)$$

with  $F_{\theta\varphi} = Q \sin \theta$ . Then, the potential is given by

$$V(\phi) = \frac{1}{2\sqrt{\phi}} \left[ 1 - \frac{Q^2}{4\phi} \right] \quad (16)$$

whose form is depicted in Fig. 1. When  $V(\phi_0) = 0$ , one has the solution to Eq. (8) as

$$\phi_0 = \frac{Q^2}{4}. \quad (17)$$

In this case, we have the  $AdS_2$  spacetime with the curvature

$$\bar{R}_2|_{r=r_e} = -V'(\phi_0) = -4/Q^3. \quad (18)$$

Finally, since the generalized entropy function is given by

$$\bar{F}^{RN}(\phi_0) = -\sqrt{\phi_0}U''(r_e), \quad (19)$$

we have the entropy for the extremal RN black hole as

$$\bar{S}_{BH}^{RN} = 4\pi\phi_0 = \pi Q^2. \quad (20)$$

As is shown in Fig. 1, one cannot find the degenerate horizon for  $Q^2 = 0$  case because it corresponds to Schwarzschild black hole.

### 4 Regular black hole

We briefly review a magnetically charged RBH [8, 9]. In this case,  $\mathcal{L}_M(B)$  in Eq. (1) is a functional of  $B = F_{\mu\nu}F^{\mu\nu}$  defined by

$$\mathcal{L}_M(B) = B \cosh^{-2} \left[ a \left( \frac{B}{2} \right)^{1/4} \right], \quad (21)$$

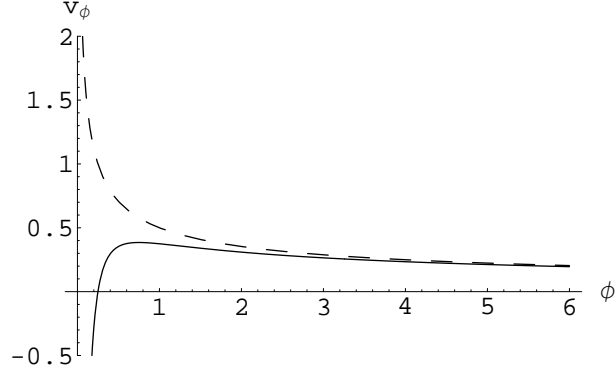


Figure 1: The solid curve: the dilaton potential  $V(\phi)$  for the RN black hole with  $Q = 1$ .  $V(\phi) = 0$  at  $\phi = \phi_0 = 0.25$  denotes the degenerate horizon. For  $Q \neq 0$ , one always finds the point of  $V(\phi) = 0$ . The large-dashed curve is for the Schwarzschild case with  $Q^2 = 0$ , where there is no point of  $V(\phi) = 0$ .

where the free parameter  $a$  will be adjusted to guarantee the regularity at the center. In the limit of  $a \rightarrow 0$ , we recover the Einstein-Maxwell theory in the previous section. To determine the metric function (2) defined by

$$U(r) = 1 - \frac{2m(r)}{r}, \quad (22)$$

we have to solve Einstein equation. From the variation of the action (1) together with the matter (21) with respect to the vector potential  $A_\mu$ , the equations of motion are given by

$$\nabla_\mu \left( \frac{d\mathcal{L}(B)}{dB} F^{\mu\nu} \right) = 0, \quad (23)$$

$$\nabla_\mu {}^* F^{\mu\nu} = 0, \quad (24)$$

where the asterisk denotes the Hodge duality. The solution to Eqs. (23) and (24) is  $F_{\theta\varphi} = Q \sin \theta$  for a magnetically charged case. On the other hand, the variation of the action with respect to the metric  $g_{\mu\nu}$  leads to the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu} \quad (25)$$

with the stress-energy tensor

$$T_{\mu\nu} = \frac{1}{4\pi} \left( \frac{d\mathcal{L}(B)}{dB} F_{\rho\mu} F_\nu^\rho - \frac{1}{4}g_{\mu\nu}\mathcal{L}(B) \right). \quad (26)$$

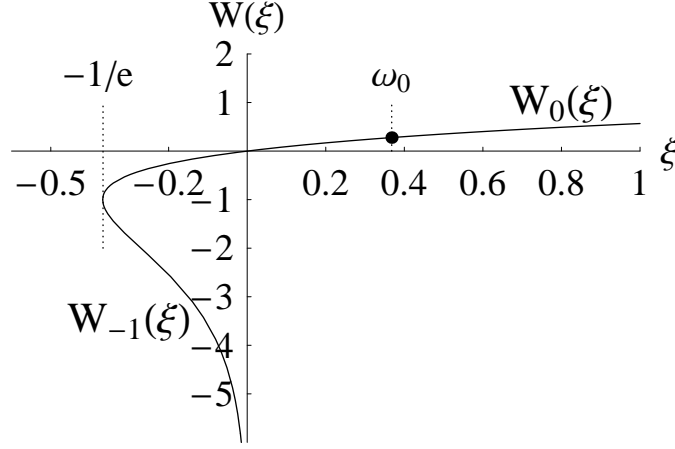


Figure 2: The two real branches of the Lambert function  $W_0(\xi)$  (upper curve) and  $W_{-1}(\xi)$  (lower curve) are depicted for the solution to the RBH. The degenerate event horizon at  $(q_e, x_e)$  corresponds to the branch point of the Lambert function at  $\xi = -1/e$ .

After solving these Einstein equation, the mass distribution is determined to be

$$m(r) = \frac{1}{4} \int^r \mathcal{L}[B(r')] r'^2 dr' + C, \quad (27)$$

where  $C$  is an integration constant. Considering the condition for the ADM mass  $M(= m(\infty) = C)$ , the mass distribution takes the form

$$m(r) = M - \frac{Q^{3/2}}{2a} \tanh\left(\frac{aQ^{1/2}}{r}\right). \quad (28)$$

Moreover, setting  $a = Q^{3/2}/2M$  determines the metric function completely as

$$U(r) = 1 - \frac{2M}{r} \left(1 - \tanh \frac{Q^2}{2Mr}\right). \quad (29)$$

At this stage we note that  $U(r)$  is regular as  $r \rightarrow 0$ , in contrast to the RN case ( $a = 0$  limit) where its metric function of  $1 - 2M/r + Q^2/r^2$  diverges as  $r^{-2}$  in that limit. In order to find the location of the horizon from  $U(r) = 0$ , we use the Lambert functions  $W_i(\xi)$  defined by the formula  $W(\xi)e^{W(\xi)} = \xi$  [8]. As is shown in Fig. 2,  $W_0(\xi)$  and  $W_{-1}(\xi)$  are real branches. Their values at branch point  $\xi = -1/e$  are the same as  $W_0(-1/e) = W_{-1}(-1/e) = -1$ . Here we set  $W_0(1/e) \equiv \omega_0$  because it plays an important role in finding the location of degenerate horizon of the extremal RBH. We note that the mass  $M$  is a free parameter. Introducing a reduced radial coordinate  $x = r/M$



and a charge-to-mass ratio  $q = Q/M$ , the condition for the event horizon is given by

$$U(x(q)) = 1 - \frac{2}{x} \left( 1 - \tanh \frac{q^2}{2x} \right) = 0. \quad (30)$$

Here one finds the outer  $x_+$  and inner  $x_-$  horizons as

$$x_+(q) = -\frac{q^2}{W_0(-\frac{q^2 e^{q^2/4}}{4}) - q^2/4}, \quad x_-(q) = -\frac{q^2}{W_{-1}(-\frac{q^2 e^{q^2/4}}{4}) - q^2/4}. \quad (31)$$

For  $q^2/4 = q_e^2/4 = w_0$ , the two horizons  $x_+$  and  $x_-$  merge into a degenerate event horizon<sup>1</sup> at

$$x_e = \frac{4q_e^2}{4 + q_e^2} = \frac{4w_0}{1 + w_0}, \quad (32)$$

where we use the relation of  $(q_e^2/4)e^{q_e^2/4} = 1/e = w_0 e^{w_0}$ . That is, the degenerate event horizon appears at  $(q_e = 1.056, x_e = 0.871)$  when  $x_+ = x_- = x_e$ . We note that in finding the location of the degenerate horizon, first we choose  $q = q_e$  and then determine  $x = x_e$ . For  $q > q_e$ , there is no horizon, while for  $q < q_e$ , two horizons appear. For our purpose, let us define the Bekenstein-Hawking entropy for the magnetically charged extremal RBH as

$$S_{BH} = \pi r_e^2 = \pi M^2 x_e^2 = \pi Q_e^2 \left[ \frac{4q_e}{4 + q_e^2} \right]^2 \quad (33)$$

with  $Q_e = Mq_e$ .

## 5 Entropy function approach

The magnetically charged extremal RBH is an interesting object because its near horizon geometry is given by the topology of  $AdS_2 \times S^2$  and its action is already known. Let us attempt to derive the black hole entropy in Eq. (33) using Sen's entropy function approach. For this purpose, we consider

---

<sup>1</sup>The near horizon geometry of the degenerate horizon  $U(r) \simeq h(r - r_e)^2$  with  $U'(r_e) = 0$  and  $U''(r_e) = 2h$ . Introducing new coordinates  $r = r_e + \varepsilon/(hy)$  and  $\tilde{t} = t/\varepsilon$  with  $h = \frac{(1+\omega_e)^3}{32M^2\omega_e^2}$ . Expanding the function  $U(r)$  in terms of  $\varepsilon$ , retaining quadratic terms and subsequently taking the limit of  $\varepsilon \rightarrow 0$ , the line element [8] becomes  $ds_{NH}^2 \simeq \frac{1}{hy^2} (-d\tilde{t}^2 + dy^2) + r_e^2 d\Omega_2^2$ . Moreover, using the Poincaré coordinate  $y = 1/u$ , one could rewrite the above line element as the standard form of  $AdS_2 \times S^2$ :  $ds_{NH}^2 \simeq \frac{1}{h} (-u^2 d\tilde{t}^2 + \frac{1}{u^2} du^2) + r_e^2 d\Omega_2^2$ .

an extremal black hole solution whose near horizon geometry is given by  $AdS_2 \times S^2$  with the magnetically charged configuration

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = v_1 \left( -r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 d\Omega_2^2, \quad (34)$$

$$F_{\theta\phi} = Q \sin \theta, \quad (35)$$

where  $v_i (i = 1, 2)$  are constants to be determined. Now, let us define the Lagrangian density  $f(v_i, Q)$  as the remaining part after integrating the action (1) over  $S^2$  as follows [26]:

$$f(v_i, Q) = \frac{1}{16\pi} \int d\theta d\phi \sqrt{-g} [R - \mathcal{L}_M(B)]. \quad (36)$$

Since  $R = -\frac{2}{v_1} + \frac{2}{v_2}$  and  $B = \frac{2Q^2}{v_2^2}$ , we obtain

$$f(v_i, Q) = \frac{1}{2} v_1 v_2 \left[ -\frac{1}{v_1} + \frac{1}{v_2} - \frac{Q^2}{v_2^2} \cosh^{-2} \left( \frac{Q^2}{2M\sqrt{v_2}} \right) \right]. \quad (37)$$

Here, we choose the free parameter  $a = Q^{3/2}/2M$  to have a RBH solution. For the magnetically charged extremal RBH, the entropy function is given by

$$\mathcal{F}(v_i, Q) = -2\pi f(v_i, Q). \quad (38)$$

In this case, the extremal values  $v_i^e$  are determined by extremizing the function  $\mathcal{F}(v_i, Q)$  with respect to  $v_i$ :

$$\frac{\partial \mathcal{F}}{\partial v_1} = 0 \Rightarrow v_2 = Q^2 \cosh^{-2} \left[ \frac{Q^2}{2M\sqrt{v_2}} \right], \quad (39)$$

$$\frac{\partial \mathcal{F}}{\partial v_2} = 0 \Rightarrow \frac{1}{v_1} = \frac{Q^2}{v_2^2} \cosh^{-2} \left[ \frac{Q^2}{2M\sqrt{v_2}} \right] - \frac{Q^2}{v_2} \frac{\partial}{\partial v_2} \left( \cosh^{-2} \left[ \frac{Q^2}{2M\sqrt{v_2}} \right] \right). \quad (40)$$

Actually, these are conventional attractor equations. Using the above relations, the entropy function at the extremum is given by

$$\mathcal{F}(v_2^e, Q) = \pi v_2^e. \quad (41)$$

In order to find the extremal value of  $v_2^e$ , we introduce  $Q = \tilde{q}M$  and  $v_2^e = M^2 \tilde{x}^2$ . Then, Eqs. (39) and (40) can be rewritten as the following equations

$$\frac{\tilde{x}^2}{\tilde{q}^2} = \cosh^{-2} \left( \frac{\tilde{q}^2}{2\tilde{x}} \right), \quad (42)$$

$$\frac{1}{v_1} = \frac{\tilde{q}^2}{M^2 \tilde{x}^4} \cosh^{-2}(\tilde{q}^2/2\tilde{x}) - \frac{\tilde{q}^4}{2M^2 \tilde{x}^5} \frac{\sinh(\tilde{q}^2/2\tilde{x})}{\cosh^3(\tilde{q}^2/2\tilde{x})}, \quad (43)$$

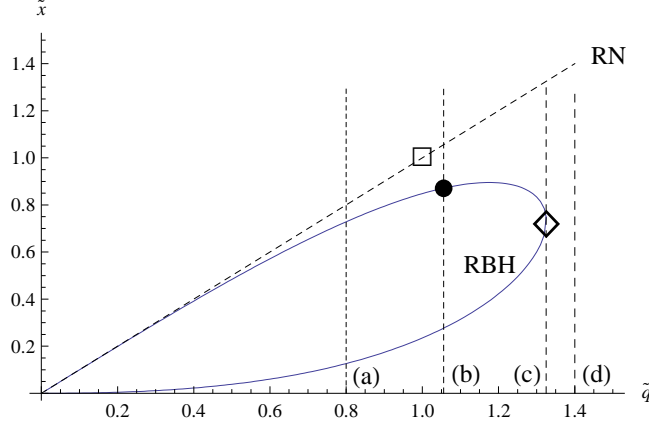


Figure 3: Plot of the curvature radius  $\tilde{x}$  of  $S^2$  versus the parameter  $\tilde{q}$ . The two curves denote the solution space of the near horizon geometry to Eq. (42), while the line denotes  $\tilde{x} = \tilde{q}$  for the RN case with its extremal point  $\tilde{x} = \tilde{q} = 1$  (□). The upper curve includes a point of ●, which corresponds to an extremal RBH. However, the lower curve belongs to the unphysical solution space because of negative  $v_1$ . ◊ denotes the critical point at  $(\tilde{q}_c, \tilde{x}_c)$ . (a)-(d) are introduced to connect the 2D dilaton potential in Fig. 4.

which are identical to the Einstein equation in the near horizon geometry of the  $AdS_2 \times S^2$  in Ref. [8] except being replaced  $1/v_1$  by  $h$ . This means that the entropy function approach is equivalent to solving the Einstein equation on the  $AdS_2 \times S^2$  background (the attractor equations). Of course, Eqs. (39) and (40) are not the full Einstein equation in (25).

Since the above equations are nonlinear differential equations, we could not solve them analytically. In the limit of  $a \rightarrow 0$ , one easily finds the extremal RN case such that  $v_2^e = M^2 \tilde{q}^2 (\tilde{x} = \tilde{q})$ ,  $v_1^e = v_2^e = Q^2$ . Instead, we have to solve numerically the Eq. (42) because of the nonlinearity between  $\tilde{x}$  and  $\tilde{q}$ . Their solutions are depicted in Fig. 3. In this figure, the solid line corresponds to the solution space in which each set of  $(\tilde{q}, \tilde{x})$  resides on the subspace  $S^2$ . There are two branches: the upper and lower ones which merge at the critical point of  $(\tilde{q}_c, \tilde{x}_c) = (1.325, 0.735)$ . Since the lower branch gives us negative  $v_1$  and thus it becomes de Sitter space instead of  $AdS_2$ , this branch should be ruled out. Note that the magnetically charged extremal RBH corresponds to the point  $(\tilde{q}_e, \tilde{x}_e) = (1.056, 0.871)$ . However, there is no way to arrive at this point even though the solution space comprises such a point. In this case, the entropy function takes the form

$$\mathcal{F} = \pi v_2^e = \pi M^2 \tilde{x}^2 = \pi M^2 \tilde{q}^2 \cosh^{-2} \left( \frac{\tilde{q}^2}{2\tilde{x}} \right). \quad (44)$$

We note that this entropy function depends on both  $\tilde{q}$  and  $\tilde{x}$ , in contrast to the RN case of  $\mathcal{F}_{RN} = \pi M^2 \tilde{x}^2 = \pi Q^2$ . Hence, unless one knows  $\tilde{q} = q_e$  and  $\tilde{x} = x_e$ , we cannot obtain the Bekenstein-Hawking entropy of  $S_{BH} = \pi M^2 x_e^2$  in Eq. (33).

## 6 Generalized entropy function approach

Before performing the conformal transformation, the entropy formula based on Wald's Noether charge formalism [20] takes the form

$$S_{BH} = \frac{4\pi}{U''(r_e)} (q_e - F(r_e)), \quad (45)$$

where the generalized entropy function  $F$  is given by

$$F(r_e) = \frac{1}{16\pi} \int_{r=r_e} d\theta d\varphi \sqrt{-g} [R - \mathcal{L}_M] \quad (46)$$

with

$$R = -\frac{r^2 U'' + 4rU' + 2U - 2}{r^2}, \quad (47)$$

$$\mathcal{L}_M = \frac{2Q^2}{r^4} \cosh^{-2} \left[ \frac{Q^2}{2Mr} \right]. \quad (48)$$

In this approach, one has to know the location of the degenerate horizon (the solution to full Einstein equations). After the integration over the angular coordinates, the generalized entropy function leads to

$$F(r_e) = \frac{1}{4} \left[ -r^2 U''(r) + 2 - r^2 \mathcal{L}_M \right] \Big|_{r=r_e} = -\frac{1}{4} U''(r_e) r_e^2 \quad (49)$$

because of  $\mathcal{L}_M|_{r=r_e} = \frac{2}{r_e^2}$  and  $U(r_e) = U'(r_e) = 0$ . Finally, for  $e = 0$  we obtain the correct form of the entropy from Eq. (45) as

$$S_{BH} = -\frac{4\pi}{U''(r_e)} F(r_e) = \pi r_e^2. \quad (50)$$

Even though we find the Bekenstein-Hawking entropy using the generalized entropy formula, there is still no way to fix the location  $r = r_e$  of the degenerate horizon. Hence we have to find another approach to calculate the entropy of an extremal RBH naturally.

## 7 2D dilaton gravity approach

Now, the remaining issue is how to incorporate the full equations of motion to extremizing process to find the entropy of an extremal RBH. We start with the action of the 2D dilation gravity in Eq. (6)

$$\bar{I}_{RBH} = \int d^2x \sqrt{-\bar{g}} [\phi \bar{R}_2 + V(\phi)] = \int d^2x \sqrt{-\bar{g}} \bar{F}, \quad (51)$$

where the Ricci scalar and the dilaton potential are

$$\bar{R}_2 = -\frac{U''}{\sqrt{\phi}}, \quad V(\phi) = \frac{1}{2\sqrt{\phi}} - \frac{Q^2}{8\phi^{3/2}} \cosh^{-2} \left[ \frac{Q^2}{4M\sqrt{\phi}} \right], \quad (52)$$

respectively. The two equations of motion are obtained as

$$\nabla^2 \phi = V(\phi), \quad (53)$$

$$\bar{R}_2 = -V'(\phi), \quad (54)$$

which give the equations for a new attractor. The solution to these equations provides the ground state for the  $AdS_2$ -gravity of Jackiw-Teitelboim theory [25]. Without any gauge-fixing, the solution to Eq. (53) may be a constant dilation  $\phi_0$  when  $V(\phi_0) = 0$ .  $V(\phi_0) = 0$  implies

$$\phi_0 = \frac{Q^2}{4} \frac{1}{\cosh^2 \left[ \frac{Q^2}{4M\sqrt{\phi_0}} \right]}. \quad (55)$$

However, this is equivalent to Eq. (42), which is one of the attractor equations in the Sen's entropy function approach. As is shown in Fig. 4, there exists a point of  $V(\phi_0) = 0$  if  $Q \leq Q_c = 1.325$ , where  $Q_c$  corresponds to the critical point  $(\tilde{x}_c, \tilde{q}_c)$ . Hence,  $V(\phi_0) = 0$  is simply another representation to the attractor equation (42). On the other hand, one may expect that the solution to Eq. (54) gives us some information on the location of the degenerate horizon. It can be rewritten as

$$U''(r)|_{r=r_e} = \frac{2Q^2}{16\phi_0^2} \frac{1}{\cosh^2 \left( \frac{Q^2}{4M\sqrt{\phi_0}} \right)} - \frac{Q^4}{32M\phi_0^{5/2}} \frac{\sinh \left( \frac{Q^2}{4M\sqrt{\phi_0}} \right)}{\cosh^3 \left( \frac{Q^2}{4M\sqrt{\phi_0}} \right)}, \quad (56)$$

which is unfortunately nothing but another attractor equation (43). Hence, it seems difficult to determine the location of the degenerate horizon of an extremal RBH using the conventional attractor equations of (55) and (56).

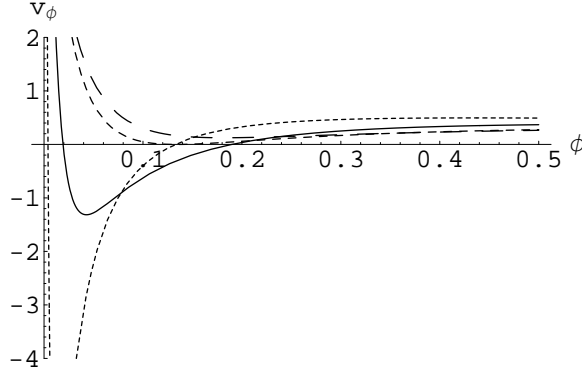


Figure 4: The solid curve: the dilaton potential  $V(\phi)$  for the extremal RBH with  $M = 1, Q_e = 1.056$  ((b) in Fig. 3).  $V(\phi_0) = 0$  is at  $\phi_0 = 0.019$  (unphysical) and  $\phi_0 = 0.19$  ( $r_e = 0.87$ ) where the latter denotes the degenerate horizon. The large-dashed curve is for no horizon with  $Q = 1.4$  ((d) in Fig. 3) where there is no point of  $V(\phi) = 0$ . The small-dashed curve is for the critical case with  $Q = 1.325$  ((c) in Fig. 3), which implies one point of  $V(\phi_0) = 0$ . The dotted curve is for another extremal black hole with  $Q = 0.8$  ((a) in Fig. 3).

In order to find the location of the degenerate horizon, we have to find the general solution to Eqs. (53) and (54) by choosing a conformal gauge of  $g_{tx} = 0$  as [27]

$$\frac{d\phi}{dx} = 2(J(\phi) - \mathcal{C}), \quad (57)$$

$$ds^2 = -(J(\phi) - \mathcal{C})dt^2 + \frac{dx^2}{J(\phi) - \mathcal{C}}, \quad (58)$$

where the 2D mass function  $J(\phi)$  is given by

$$J(\phi) = \int^\phi V(\tilde{\phi})d\tilde{\phi} = \sqrt{\phi} + M \tanh\left(\frac{Q^2}{4M\sqrt{\phi}}\right). \quad (59)$$

Here  $\mathcal{C}$  is a coordinate-invariant constant of the integration, which is identified with the mass  $M$  of the extremal black hole. A necessary condition that a 2D dilaton gravity admits an extremal RBH is that there exists at least one curve of  $\phi = \phi_0 = \text{const}$  such that  $J(\phi) = M$ . Actually, we have an important relation between the 4D metric function  $U(r)$  and 2D mass function  $J(\phi)$  as

$$\sqrt{\phi}U(r(\phi)) = J(\phi) - M \quad (60)$$

with  $r = 2\sqrt{\phi}$ . In addition,  $J(\phi)$  should be monotonic in a neighborhood of  $\phi_0$  with the attractor conditions  $J'(\phi_0) = V(\phi_0) = 0$  and  $J''(\phi_0) = V'(\phi_0) \neq 0$

to have the extremal black hole. These correspond to the attractor conditions in Eq. (10). First,  $J(\phi) = M$  determines the horizons  $r = r_{\pm}$

$$U(\phi_{\pm}) = 1 - \frac{M}{\sqrt{\phi_{\pm}}} \left[ 1 - \tanh \left( \frac{Q^2}{4M\sqrt{\phi_{\pm}}} \right) \right] = 0 \rightarrow U(r_{\pm}) = 0. \quad (61)$$

Considering the connection of  $\phi_0 = \frac{1}{4}r_e^2$ , the attractor conditions of  $J'(\phi_0) = 0$  and  $J''(\phi_0) \neq 0$  implies the extremal configuraion

$$U'(r_e) = 0, \quad U''(r_e) \neq 0. \quad (62)$$

Then, combining Eq. (61) with Eq. (62) leads to the condition for finding the degenerate horizon  $r = r_e$ . Following Sec. 4, for  $Q_e = Mq_e = 2\sqrt{w_0}M$ , we find the location of the degenerate horizon,  $r_e = Mx_e = 4Mw_0/(1 + w_0)$ . Here, we have the  $AdS_2$  spacetime with negative constant curvature

$$\bar{R}_2|_{r=r_e} = -\frac{2h}{\sqrt{\phi_0}} = -\frac{1}{\sqrt{\phi_0}}U''(r_e) = -\frac{(1 + \omega_0)^4}{16M^3\omega_0^3}. \quad (63)$$

The generalized entropy function takes the form

$$\bar{F}^{RBH}(\phi_0) = -\sqrt{\phi_0}U''(r_e). \quad (64)$$

Finally, for the magnetically charged extremal RBH, the desired Bekenstein-Hawking entropy is given by

$$\bar{S}_{BH}^{RBH} = -\frac{4\pi\sqrt{\phi_0}}{U''(r_e)}\bar{F}^{RBH}(\phi_0) = 4\pi\phi_0 = \pi r_e^2. \quad (65)$$

Given  $\tilde{x} = x_e$ ,  $\tilde{q} = q_e$ , this entropy can be exactly recovered from Eq. (44) in the entropy function approach as

$$\mathcal{F}(x_e, q_e) = \pi M^2 q_e^2 \cosh^{-2} \left( \frac{q_e^2}{2x_e} \right) = \pi r_e^2 = \pi M^2 x_e^2. \quad (66)$$

Here, we also note that the  $AdS_2$  curvature  $\bar{R}_2(U''(r_e))$  and  $\sqrt{\phi_0}$  are irrelevant to determining the entropy of the extremal RBH. Furthermore, we confirm that the entropy is invariant under the conformal transformation because  $\sqrt{\phi_0}$  is a conformal factor [20].

## 8 Discussions

We have discussed a magnetically charged RBH derived from the coupled action of Einstein gravity and nonlinear electrodynamics. Although the action is simple, it is very interesting to investigate its extremal black hole because its action is nonlinear on the Maxwell side. This black hole solution is parameterized by the ADM mass  $M$  and magnetic charge  $Q$ , while the free parameter  $a = Q^{3/2}/2M$  is adjusted to make the resultant line element regular at the center. It turned out that the entropy function approach does not lead to a correct entropy of the Bekenstein-Hawking entropy. This is mainly because the magnetically charged extremal RBH is obtained from the coupled system of Einstein gravity and nonlinear electrodynamics. In the limit of  $a \rightarrow 0$  (Einstein-Maxwell theory), one finds the condition of  $M = Q$  for the RN black hole. In this case, all approaches mentioned by this work provide the Bekenstein-Hawking entropy  $S_{BH}^{RN} = \pi Q^2$  because of its linearity  $\tilde{x} = \tilde{q}$ , where  $r = M\tilde{x}$  and  $Q = M\tilde{q}$ . For  $a = Q^{3/2}/2M$  case of the extremal RBH, there is no linearity between  $\tilde{x}$  and  $\tilde{q}$  and instead, their connection is determined by the nonlinearity of  $\tilde{x} = \tilde{q} \cosh^{-1}(\tilde{q}^2/2\tilde{x})$  in Eq. (42). It follows that the entropy function approach is sensitive to whether the nature of the central region of the black hole is regular or singular.

The two attractor equations in Eqs. (39) and (40) are not enough to determine the entropy of the extremal RBH because we have a lot of solutions satisfying these same attractor equations in Fig. 3. That is, Eq. (39) of  $\tilde{x} = \tilde{q} \cosh^{-1}(\tilde{q}^2/2\tilde{x})$  does not imply the condition for determining the degenerate horizon of  $U(x) = U'(x) = 0$ ,  $U''(x) \neq 0$ . Solving the Einstein equations in the near horizon geometry is not sufficient to obtain the entropy of the extremal RBH. Hence, to find the correct form of the entropy of extremal black hole, we introduce the generalized entropy formula combined with a 2D dilaton gravity. In this case, the new attractor equations are given by Eqs. (53) and (54), which contain full information on the location of the degenerate horizon. Using the 2D dilation gravity approach, the new attractor equations provide the condition of (61) and (62) for determining the location of the degenerate horizon. Also we check that Eq. (39) is satisfied with  $\tilde{x} = x_e$  and  $\tilde{q} = q_e$ , corresponding to  $\bullet$  in Fig. 3.

At this stage, we would like to mention the difference between the RN and RBH black holes in obtaining entropies of their extremal black holes. In the case of the RN black hole, the attractor equation of  $\tilde{x} = \tilde{q}(r = Q)$  with the free parameter  $M$  is enough to determine the Bekenstein-Hawking entropy as the extremal black hole entropy. This means that the extremal condition of  $\tilde{x} = \tilde{q} = 1(r = M = Q)$  is not necessary for finding the extremal entropy. However, for the RBH with the free parameter  $M$ , we have to know



the extremal position of  $x_e$  and the charge  $q_e$  exactly to obtain the entropy because the attractor equation of  $\tilde{x} = \tilde{q} \cosh^{-1}(\tilde{q}^2/2\tilde{x})$  is nonlinear.

In this work, we have succeeded to find the entropy of the extremal RBH by using the 2D dilaton gravity approach. This approach provides the location of horizon with attractor conditions for degenerate horizon, which are  $U(x_e) = U'(x_e) = 0, U''(x_e) \neq 0$ . We stress that this is not possible if one does not use the dilaton gravity approach known as the  $s$ -wave approximation of 4D gravity theory. Using Sen's entropy function approach, one can get  $U'(x_e) = 0$  and  $U''(x_e) \neq 0$  partly. In other words, although Sen's entropy function approach is known to provide Einstein equation in the near horizon geometry of  $AdS_2$  spacetime as the attractor equations, this does not work for the regular black hole. Formally, we have to use full Einstein equations to find the entropy of extremal regular black hole. However, noting that the  $s$ -wave approximation preserves the attractor conditions, we have used the 2D dilaton gravity approach to find the location of the degenerate horizon for simplicity, instead of solving the full Einstein equations.

In conclusion, we have successfully obtained the entropy of an extremal regular black hole from the generalized entropy formula based on Wald's Noether charge formalism combined with the 2D dilaton gravity approach.

## Acknowledgments

We would like to thank L.-M. Cao for useful discussions. Two of us (Y. S. Myung and Y.-J. Park) were supported by the Science Research Center Program of the Korea Science and Engineering Foundation through the Center for Quantum Spacetime of Sogang University with grant number R11-2005-021. Y. S. Myung was also in part supported by the Korea Research Foundation (KRF-2006-311-C00249) funded by the Korea Government (MOEHRD). Y.-W. Kim was supported by the Korea Research Foundation Grant funded by Korea Government (MOEHRD): KRF-2007-359-C00007.

## References

- [1] P. Ginsparg and M. J. Perry, Nucl. Phys. B **222**, 245 (1983); O. B. Zaslavskii, Phys. Rev. D **56**, 2188 (1997); M. Caldarelli, L. Vanzo, and S. Zerbini, [arXiv:hep-th/0008136]; O. J. C. Dias and J. P. S. Lemos,

- Phys. Rev. D **68**, 104010 (2003); V. Cardoso, O. J. C. Dias, and J. P. S. Lemos, Phys. Rev. D **70**, 024002 (2004).
- [2] J. Bardeen, “Non-singular general-relativistic gravitational collapse,” In Proceedings of GR5, Tiflis, U.S.S.R, p174 (1968); J. Bardeen, Phys. Rev. Lett. **46**, 382 (1981).
- [3] A. Borde, Phys. Rev. D **50**, 3692 (1994); E. Ayon-Beato and A. Garcia, Phys. Rev. Lett. **80**, 5056 (1998).
- [4] I. Dymnikova, Gen. Rel. Grav. **24**, 235 (1992).
- [5] E. Ayon-Beato and A. Garcia, Phys. Lett. B **464**, 25 (1999); S. A. Hayward, Phys. Rev. Lett. **96**, 031103 (2006); A. Bonanno and M. Reuter, Phys. Rev. D **73**, 083005 (2006); T. G. Rizzo, JHEP **0609**, 021 (2006); P. Nicolini, A. Smailagic, and E. Spallucci, Phys. Lett. B **632**, 547 (2006); S. Ansoldi, P. Nicolini, A. Smailagic, and E. Spallucci, Phys. Lett. B **645**, 261 (2007); Y. S. Myung, Y.-W. Kim, and Y.-J. Park, JHEP **0702**, 012 (2007); Y. S. Myung, Y.-W. Kim, and Y.-J. Park, [arXiv:gr-qc/0702145];.
- [6] K. A. Bronnikov, Phys. Rev. D **63**, 044005 (2001).
- [7] J. Matyjasek, Phys. Rev. D **63**, 084004 (2001).
- [8] J. Matyjasek, Phys. Rev. D **70**, 047504 (2004).
- [9] W. Berej, J. Matyjasek, D. Tryniecki, and M. Woronowicz, Gen. Rel. Grav. **38**, 885 (2006).
- [10] B. Zwiebach, Phys. Lett. B **156**, 315 (1985); C. G. Callan, I. R. Klebanov, and M. J. Perry, Nucl. Phys. B **278**, 78 (1986); D. J. Gross and E. Witten, Nucl. Phys. B **277**, 1 (1986).
- [11] C. G. Callan, R. C. Myers, and M. J. Perry, Nucl. Phys. B **311**, 673 (1989); R. C. Myers, [arXiv:gr-qc/9811042].
- [12] B. de Wit, Fortsch. Phys. **54**, 183 (2006); T. Mohaupt, [arXiv:hep-th/0512048].
- [13] R. M. Wald, Phys. Rev. D **48**, 3427 (1993); T. Jacobson, G. Kang, and R. C. Myers, Phys. Rev. D **49**, 6587 (1994); V. Iyer and R. M. Wald, Phys. Rev. D **50**, 846 (1994); T. Jacobson, G. Kang, and R. C. Myers, [arXiv:gr-qc/9502009].

- [14] K. Behrndt, G. Lopes Cardoso, B. de Wit, D. Lust, T. Mohaupt, and W. A. Sabra, Phys. Lett. B **429**, 289 (1998); A. Dabholkar, R. Kallosh, and A. Maloney, JHEP **0412**, 059 (2004); A. Sen, JHEP **0505**, 059 (2005); D. Bak, S. Kim, and S. J. Rey, [arXiv:hep-th/0501014].
- [15] K. Goldstein, N. Iizuka, R. P. Jena, and S. P. Trivedi, Phys. Rev. D **72**, 124021 (2005).
- [16] M. Alishahiha and H. Ebrahim, JHEP **0603**, 003 (2006); R. Kallosh, N. Sivanandam, and M. Soroush, JHEP **0603**, 060 (2006); A. Sinha and N. V. Suryanarayana, Class. Quant. Grav. **23**, 3305 (2006); B. Chandrasekhar, S. Parvizi, A. Tavanfar, and H. Yavartanoo, JHEP **0608**, 004 (2006); D. Astefanesei, K. Goldstein, and S. Mahapatra, [arXiv:hep-th/0611140].
- [17] A. Sen, JHEP **0509**, 038 (2005); A. Sen, JHEP **0603**, 008 (2006).
- [18] B. Sahoo and A. Sen, JHEP **0607**, 008 (2006); B. Chandrasekhar, [arXiv:hep-th/0604028]; R. G. Cai and D. W. Pang, Phys. Rev. D **74**, 064031 (2006); R. G. Cai and D. W. Pang, [arXiv:hep-th/0702040]; S. Hyun, W. Kim, J.J. Oh, and E.J. Son, JHEP **0704**, 057 (2007); J.-H. Cho and S. Nam, JHEP **0706**, 011 (2007).
- [19] H. K. Kunduri, J. Lucietti, and H.S. Reall, [arXiv:0705.4214(hep-th)]; D. Astefanesei and H. Yavartanoo, [arXiv:0706.1847(hep-th)].
- [20] R.-G. Cai and L.-M. Cao, [arXiv:0704.1239(hep-th)] (Private Communication).
- [21] Y. S. Myung, Y.-W. Kim, and Y.-J. Park, [arXiv:0705.2478(gr-qc)].
- [22] J. Navarro-Salas and P. Navarro, Nucl. Phys. B **579**, 250 (2000).
- [23] A. Fabbri, D. J. Navarro, and J. Navarro-Salas, Nucl. Phys. B **595**, 381 (2001).
- [24] D. Grumiller, W. Kummer, and D. V. Vassilevich, Phys. Rept. **369**, 327 (2002); D. Grumiller and R. McNees, JHEP **0704**, 074 (2007).
- [25] R. Jackiw, in S. M. Christensen, Quantum Theory of Gravity, Hilger, Bristol, p.403 (1984); C. Teitelboim, in S. M. Christensen, Quantum Theory of Gravity, Hilger, Bristol, p.327 (1984).
- [26] A. Sen, JHEP **0507**, 073 (2005).

- [27] J. Gegenberg, G. Kunstatter, and D. Louis-Martinez, Phys. Rev. D **51**, 1781 (1995); J. Cruz, A. Fabbri, D. J. Navarro, and J. Navarro-Salas, Phys. Rev. D **61**, 024011 (2000).